Topology Classes of Flat U(1) Bundles and Diffeomorphic Covariant Representations of the Heisenberg Algebra

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Abstract

The general construction of self-adjoint configuration space representations of the Heisenberg algebra over an arbitrary manifold is considered. All such inequivalent representations are parametrised in terms of the topology classes of flat U(1) bundles over the configuration space manifold. In the case of Riemannian manifolds, these representations are also manifestly diffeomorphic covariant. The general discussion, illustrated by some simple examples in non relativistic quantum mechanics, is of particular relevance to systems whose configuration space is parametrised by curvilinear coordinates or is not simply connected, which thus include for instance the modular spaces of theories of non abelian gauge fields and gravity.

PACS numbers: 02.30.-f, 03.65.-w

August 1999

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1 Introduction

The representation theory of the Heisenberg algebra is of fundamental importance to the canonical quantisation programme of Lagrangian and Hamiltonian systems. Indeed, through the correspondence principle[1], the symplectic structure of the classical phase space determines the algebra of (anti)commutation relations of quantum observables. In the case of conjugate pairs of Grassmann even canonical phase space variables (real under complex conjugation), say q and p, the quantum observables $\hat{q}$ and $\hat{p}$ are defined by the commutation relation

$$[\hat{q}, \hat{p}] = i\hbar \quad ,$$

as well as by the self-adjoint properties $\hat{q}^\dagger = \hat{q}$ and $\hat{p}^\dagger = \hat{p}$. This set of conditions defines the Heisenberg algebra in the case of a single set of canonical conjugate observables.

As is well known[2], when the configuration space variable q takes its values in the entire real line, there exists essentially only one representation of the algebra, up to unitary transformations, provided by the usual plane wave functions. However, going beyond that simplest case (or trivial extensions of it obtained by direct products) has proven difficult[1], and it appears that no general procedure has been put forward in the case of an arbitrary connected manifold, or even more simply when curvilinear coordinates are used in euclidean spaces.

In the present paper, an approach to this problem is suggested, which, by combining algebraic and topological arguments, leads to a general classification of the inequivalent representations of the Heisenberg algebra in terms of certain topological properties of the underlying manifold, namely those properties which may be characterized through a flat U(1) bundle over it. Moreover, when the manifold is endowed with a Riemannian metric, the corresponding representations are manifestly covariant under the diffeomorphism group of coordinate transformations, thereby including curvilinear parametrisations of configuration space. Finally, the requirement of the self-adjoint properties of the algebra, and beyond it, of the quantum dynamics, leads to clearly identified specific restrictions on wave function representations.

The present discussion has no pretense to mathematical rigour, but rather is a physicist's constructive approach, deferring the more subtle issues of functional analysis to a later stage. As such, it also is a clear invitation to mathematical physicists to dwelve further into the mathematical justifications beyond the settings of the problem as advocated here.

The paper is organised as follows. The next section develops the representation theory of the Heisenberg algebra for an arbitrary discrete number of conjugate pairs of canonical variables, requiring only algebraic and topological considerations. This is followed by Sect. 3 which elaborates somewhat further on configuration and momentum space wave function representations. Sect. 4 then applies that discussion to a more physical setting, at which stage geometrical and quantum dynamical aspects come into play. These three sections thus detail our main general results, which are illustrated in Sects.5 and 6 through some simple examples borrowed from non relativistic quantum mechanics. Finally, further remarks are presented in the Conclusions.

2 The Abstract Setting: Algebra and Topology

Quite generally, let us consider a discrete set of real classical conjugate observables (q^α, p_α) $(\alpha = 1, 2, \ldots, n)$. The number n of such pairs, if infinite, could be discrete or non countable, thereby raising further issues of mathematical justifications. For definiteness though, we shall think of that number n as being finite in the present discussion.

Moreover, at the present stage, the variables q^α and p_α are not necessarily a set of coordinates of the configuration space of a physical system, and of their conjugate momenta

[1]Even the case of the positive real line has been much discussed[2, 3].

parametrising the cotangent bundle. Indeed, they could correspond to a collection of conjugate observables which, at the quantum level, are assumed to obey the Heisenberg algebra. Nevertheless, since such a situation is certainly always encountered for the fundamental canonical phase space degrees of freedom of a dynamical system, and as such must thus always be addressed, again for definiteness we shall think of the variables q^α as parametrising[2] a specific but otherwise arbitrary connected[3] differentiable manifold M of dimension n. Except for its connectedness, no further properties of the manifold are assumed, such as compactness or the existence of boundaries, or even a Riemannian metric structure compatible with its topology. Such additional specifications will stem from the geometrical and physical properties of a specific system to be quantised, and are thus considered only in Sect.4.

Let us remark that the rôle of the variables q^α is thus distinguished from the outset from that of their conjugate momenta p_α, the q^α being the coordinates which parametrise the configuration space of a given system. The domain of values for q^α is thus specified by the choice of manifold M, *i.e.* of physical system, while that of their conjugate momenta p_α is determined *at the quantum level* by the representation theory of their Heisenberg algebra, the latter thus being defined by the relations

$$[\hat{q}^\alpha, \hat{p}_\beta] = i\hbar\, \delta^\alpha_\beta \quad , \quad (\hat{q}^\alpha)^\dagger = \hat{q}^\alpha \quad , \quad (\hat{p}_\alpha)^\dagger = \hat{p}_\alpha \quad , \quad \alpha, \beta = 1, 2, \ldots, n \quad . \tag{1}$$

Indeed, even though the Heisenberg algebra may appear to be dual under the combined exchange of the conjugate variables $\hat{q}^\alpha \leftrightarrow \hat{p}_\alpha$ and a sign reversal of $\hbar$, possibly different domains of spectral values for the operators $\hat{q}^\alpha$ and $\hat{p}_\alpha$ render this apparent duality in general meaningless[4].

In order to develop an abstract representation theory of the Heisenberg algebra, only the following two assumptions are made[5], of which the first does indeed emphasize the distinguished rôle assumed in our approach by the configuration space manifold M.

A1. There exists a basis $|q>$ of the representation space which is spanned by eigenstates of the position operators $\hat{q}^\alpha$ ($\alpha = 1, 2, \ldots, n$), whose domain of eigenvalues coincides with all the values of the coordinates q^α parametrising the configuration space manifold M,

$$\hat{q}^\alpha\, |q> = q^\alpha\, |q> \quad , \quad \{q^\alpha\} \,\epsilon\, M \quad , \quad \alpha = 1, 2, \ldots, n \;\; ; \tag{2}$$

A2. The representation space of the algebra may be endowed with an hermitian positive definite inner product $< \cdot | \cdot >$ for which the operators $\hat{q}^\alpha$ and $\hat{p}_\alpha$ ($\alpha = 1, 2, \ldots, n$) are self-adjoint.

Armed with these two assumptions, we shall now determine the necessary conditions for such a general representation of the Heisenberg algebra to exist. Since, given these assumptions, the knowledge of the configuration space matrix elements of the operators $\hat{q}^\alpha$ and $\hat{p}_\alpha$ is tantamount to specifying the representation itself, let us first consider the former type of matrix element, namely $< q|\hat{q}^\alpha|q' >$. Clearly, by virtue of both assumptions, its evaluation implies the relations

$$(q^\alpha - q'^\alpha) < q|q' >= 0 \quad , \quad \alpha = 1, 2, \ldots, n \quad . \tag{3}$$

[2]Without risk of confusion, a common notation for the coordinate parametrisation of the manifold is used, even when its topology requires more than one coordinate system to cover it entirely.

[3]Were the manifold to be disconnected, each of its connected components would correspond to the configuration space of a distinct physical subsystem.

[4]Except in some fortuitous cases where they do coincide, for example when the manifold M is isomorphic to the n dimensional euclidean space.

[5]A similar approach was applied in Ref.[4] to the simple case $n = 1$ of a single degree of freedom over the real line. In the course of completing this paper, it was realised that the same point of view has already been advocated in Ref.[5], without however, pursuing the consequences of this approach to its full conclusions.

Consequently, one necessarily has the following parametrisation of the inner product expressed in the basis of position eigenstates

$$< q|q' >= \frac{1}{\sqrt{g(q)}}\,\delta^{(n)}(q-q') \quad , \tag{4}$$

where $g(q)$ is *a priori* an arbitrary positive definite function defined over the configuration space manifold M. This result implies the spectral decomposition of the identity operator in the position eigenbasis $|q>$,

$$\mathbb{1} = \int_M d^nq\sqrt{g(q)}\,|q><q| \quad , \tag{5}$$

which in turn, leads to the configuration space wave function representations $\psi(q) =< q|\psi >$ and $< \psi|q >=< q|\psi >^* = \psi^*(q)$ of any state $|\psi >$ belonging to the Heisenberg algebra representation space,

$$|\psi >= \int_M d^nq\sqrt{g(q)}\psi(q)|q> \quad , \quad <\psi| = \int_M d^nq\sqrt{g(q)}\psi^*(q)<q| \quad . \tag{6}$$

In particular, the inner product of any two states $|\psi >$ and $|\varphi >$ is given in terms of their configuration space wave functions $\psi(q)$ and $\varphi(q)$, respectively, by

$$< \psi|\varphi >= \int_M d^nq\sqrt{g(q)}\psi^*(q)\varphi(q) \quad . \tag{7}$$

Obviously, the choice of function $g(q)$ is directly related to the absolute normalisation of the position eigenbasis states $|q >$, which is entirely left unspecified by the assumptions A1 and A2 above. By choosing in (4) a parametrisation in terms of the positive square root of $g(q)$, the assumed positive definiteness of the inner product is made manifest. In addition, as is made explicit in (7), when the manifold M is endowed with a Riemannian metric $g_{\alpha\beta}(q)$, the canonical choice for the function $g(q)$ is the determinant (det $g_{\alpha\beta}(q)$), thereby justifying the form of the parametrisation used in (4). This point will be elaborated on in Sect.4.

Note however that having specified the absolute normalisation of the states $|q >$ in terms of the function $g(q)$, still does not specify their phase, which is also left entirely free by the assumptions A1 and A2. As will become clear later on, this remaining freedom in the choice of phase for the position eigenbasis plays a crucial rôle in the classification of inequivalent representations of the Heisenberg algebra.

Let us now turn to the determination of the position matrix elements of the momentum operators $\hat{p}_\alpha$. For this purpose, consider first the position matrix elements of the Heisenberg algebra (1),

$$< q|\,[\hat{q}^\alpha, \hat{p}_\beta]\,|q' >= i\hbar\delta^\alpha_\beta\,\frac{1}{\sqrt{g(q)}}\,\delta^{(n)}(q-q') \quad . \tag{8}$$

Since, using assumptions A1 and A2, the l.h.s. of this identity reduces to $(q^\alpha - q'^\alpha) < q|\hat{p}_\beta|q' >$, the relevant matrix elements may be parametrised as

$$< q|\hat{p}_\alpha|q' >= \frac{-i\hbar}{\sqrt{g(q)}}\frac{\partial}{\partial q^\alpha}\delta^{(n)}(q-q') + \frac{1}{\sqrt{g(q)}}\Big[A_\alpha(q) + iB_\alpha(q)\Big]\delta^{(n)}(q-q') \quad , \tag{9}$$

where $A_\alpha(q)$ and $B_\alpha(q)$ ($\alpha = 1, 2, \ldots, n$) are the components of two *a priori* arbitrary real vector fields defined over the manifold M.

However, these two vector fields are restricted further by the Heisenberg algebra and the assumptions A1 and A2. Indeed, the requirement of hermiticity of the inner product and the self-adjoint property of $\hat{p}_\alpha$,

$$< q|\hat{p}_\alpha|q' >^* =< q'|\hat{p}^\dagger_\alpha|q >=< q'|\hat{p}_\alpha|q > \quad , \tag{10}$$

implies the following expression for the vector field $B_\alpha(q)$

$$B_\alpha(q) = -\frac{1}{2}\hbar\sqrt{g(q)}\frac{\partial}{\partial q^\alpha}\left(\frac{1}{\sqrt{g(q)}}\right) \quad . \tag{11}$$

Consequently, (9) reduces to

$$< q|\hat{p}_\alpha|q' >= \frac{-i\hbar}{g^{1/4}(q)}\frac{\partial}{\partial q^\alpha}\left(\frac{1}{g^{1/4}(q)}\delta^{(n)}(q-q')\right) + \frac{1}{\sqrt{g(q)}}A_\alpha(q)\delta^{(n)}(q-q') \quad . \tag{12}$$

Furthermore, position matrix elements of the relations[6] $[\hat{p}_\alpha, \hat{p}_\beta] = 0$ may be evaluated using (12), leading to the following restrictions on the vector field $A_\alpha(q)$,

$$A_{\alpha\beta}(q) \equiv \frac{\partial A_\beta(q)}{\partial q^\alpha} - \frac{\partial A_\alpha(q)}{\partial q^\beta} = 0 \quad , \quad \alpha, \beta = 1, 2, \ldots, n \quad . \tag{13}$$

Obviously, this result suggests that in fact the vector field $A_\alpha(q)$ defines a section of a U(1) bundle over the configuration manifold M, which, by virtue of (13), would then have to be a flat bundle. To establish this specific characterization of the vector field $A_\alpha(q)$, let us return to the remaining freedom in phase redefinitions of the position eigenbasis $|q>$. Under an arbitrary local phase transformation of these eigenvectors[7],

$$|q>_{(2)} = e^{i/\hbar\,\chi(q)}\,|q> \quad , \tag{14}$$

where $\chi(q)$ is an arbitrary local scalar function over M, it is straightforward to show that the vector field $A^{(2)}_\alpha(q)$ associated to the matrix elements ${}_{(2)}< q|\hat{p}_\alpha|q' >_{(2)}$ is then related to the vector field $A_\alpha(q)$ associated to the matrix elements $< q|\hat{p}_\alpha|q' >$ by the transformation

$$A^{(2)}_\alpha(q) = A_\alpha(q) + \frac{\partial\chi(q)}{\partial q^\alpha} \quad . \tag{15}$$

This result thus establishes that the vector field $A_\alpha(q)$, which parametrises the position matrix elements of the momentum operators $\hat{p}_\alpha$ in (12), does indeed define a flat U(1) bundle over M. The existence of this bundle with connection $A_\alpha(q)$ is thus associated to the freedom in the choice of phase of the position eigenbasis $|q>$, while the freedom in the normalisation of these states is associated to the choice of function $g(q)$.

Note that this characterization of the vector field $A_\alpha(q)$ indeed requires that as a U(1) connection, it be globally well defined over the entire manifold M. If the topology of M is such that more than one coordinate patch is required, position eigenstates $|q>$ associated to the overlap of two patches could differ by a local phase transformation when passing from one patch to the other. Such a transformation however, then affects equally the U(1) connection $A_\alpha(q)$ in a manner consistent with the transition functions between the coordinate patches. Hence, the vector field $A_\alpha(q)$ is indeed a U(1) connection which is well defined globally over M, as is thus also the corresponding configuration space representation of the Heisenberg algebra.

To conclude so far, configuration space representations of the Heisenberg algebra over the manifold M are thus characterized, on the one hand, by the function[8] $g(q)$, and on the other hand, by a flat U(1) bundle. However, since arbitrary local gauge transformations within the U(1) bundle correspond to arbitrary local phase redefinitions of the states $|q>$, and thus

[6]The relations $[\hat{q}^\alpha, \hat{q}^\beta] = 0$ are obviously satisfied for the representations considered, since the operators $\hat{q}^\alpha$ are diagonal in the position eigenbasis $|q>$ by virtue of assumption A1.

[7]Such transformations obviously preserve the normalisation (4).

[8]Which, for a Riemannian manifold, will correspond to the determinant of the metric, hence to a geometrical structure; see Sect.4.

relate representations of the Heisenberg algebra which are unitarily equivalent, it is clear that all *inequivalent* representations of the Heisenberg algebra over a manifold M are classified in terms of the topologically distinct flat U(1) bundles over that manifold, *i.e.* the equivalence classes under *local* gauge transformations of U(1) gauge fields over M of vanishing field strength.

In particular, a trivial flat U(1) bundle corresponds to a representation of the algebra for which the gauge freedom of phase redefinitions of the eigenstates $|q>$ may be used to gauge away locally the vector field $A_\alpha(q)$ all over the manifold M, the field being then a pure gauge configuration, $A_\alpha(q) = \partial\chi(q)/\partial q^\alpha$. Otherwise, a non trivial flat U(1) bundle corresponds to a representation of the Heisenberg algebra for which there is a topological obstruction to the gauging away of the vector field $A_\alpha(q)$ globally over the entire manifold. Hence, the local algebraic characterization of any representation of the Heisenberg algebra is constrained nevertheless by global topological properties of the configuration space manifold M through the possible existence of a non trivial flat U(1) bundle over M associated to that representation of the algebra.

It is well known that the topology classes of flat U(1) bundles are characterized by their holonomies around non trivial cycles in the manifold, or more precisely[9], by the maps of the generators of the first homotopy group $\pi_1(M)$ into the gauge group U(1). If the manifold is simply connected, only a trivial flat bundle is possible, since all holonomies are then contractible to the identity. Hence over a simply connected manifold, the Heisenberg algebra admits only a single representation, given by the relations (4) and (12) with $A_\alpha(q) = 0$ (or equivalently a pure gauge, *i.e.* a gradient $A_\alpha(q) = \partial\chi(q)/\partial q^\alpha$, which simply defines a unitarily equivalent representation). For a non simply connected manifold however, topology classes of non trivial flat bundles are thus characterized by their non trivial holonomies around all non contractible cycles in configuration space.

In conclusion, unitarily inequivalent configuration space representations of the Heisenberg algebra over an arbitrary manifold M are parametrised by two types of data. On the one hand, a function $g(q)$ which determines the absolute normalisation of the position eigenbasis $|q>$. On the other hand, a flat U(1) bundle over M whose holonomies uniquely characterize the representation. The first information is of a geometrical character, since it determines an integration measure over the manifold for the configuration space representation of the inner product (see (7)). At this stage however, having not yet bestowed the manifold with a geometry, the choice for $g(q)$ is totally arbitrary and this information is thus not yet complete[10]. The second information however, is of a global topological character, is complete, and is entirely specified from the local algebraic properties of the Heisenberg algebra representation over M.

Consequently, for simply connected manifolds M, the Heisenberg algebra possesses only a single configuration space representation up to unitary transformations. This representation is characterized by the arbitrary integration measure $d^n q\sqrt{g(q)}$ and some trivial flat U(1) bundle over M, whose section $A_\alpha(q)$ may always be taken to vanish identically. Clearly, this conclusion generalises to any simply connected manifold the very same fact known for a long time in the case of the real line[2].

In the case of non simply connected manifolds however, beyond the trivial representation associated to a trivial flat U(1) bundle, the Heisenberg algebra possesses an infinity of inequivalent representations labelled by all topologically distinct non trivial flat U(1) bundles, namely by the embeddings of the generators of the mapping class group $\pi_1(M)$ into U(1) such that at least one of its non trivial homotopy class generators is mapped onto a non trivial U(1) phase.

[9]The line integrals of $A_\alpha(q)$ along two homotopically equivalent cycles differ by the integral of the field strength $A_{\alpha\beta}(q)$ over a connected surface in M bounded by the two cycles. This difference thus vanishes for a flat bundle.

[10]For a Riemaniann manifold, a natural choice for $g(q)$ is the determinant of the metric.

3 Configuration and Momentum Space Wave Functions

Having thus completely characterized the representations of the Heisenberg algebra over an arbitrary manifold M solely in terms of algebraic and topological considerations, let us momentarily return to some consequences of these conclusions, beginning with the configuration space wave functions of states, $\psi(q) =< q|\psi >$, and the ensuing position and momentum operator representations.

Given the parametrisation (12) of the momentum operator configuration space matrix elements, and the representation (5) of the unit operator, the configuration space wave function representation[11] of $\hat{p}_\alpha$ is provided by the differential operator

$$< q|\hat{p}_\alpha|\psi >= \frac{-i\hbar}{g^{1/4}(q)}\frac{\partial}{\partial q^\alpha}\left[g^{1/4}(q)\psi(q)\right] + A_\alpha(q)\,\psi(q) = \frac{-i\hbar}{g^{1/4}(q)}\left[\frac{\partial}{\partial q^\alpha} + \frac{i}{\hbar}A_\alpha(q)\right]g^{1/4}(q)\psi(q) \quad , \tag{16}$$

in which one recognizes the covariant derivative for the U(1) gauge connection $A_\alpha(q)$. This expression is thus the appropriate generalisation of the usual correspondence rule, which ignores the integration measure factors $g^{1/4}(q)$—stemming from the normalisation of the position eigenbasis $|q >$—and the possibly non trivial flat U(1) bundle $A_\alpha(q)$—stemming from the topological properties of the base manifold M and their possible obstruction to a complete gauging away of a non trivial flat U(1) connection. The usual correspondence rule for the momentum operators in the configuration space representation is thus seen to be associated to the trivial representation of the Heisenberg algebra with $A_\alpha(q) = 0$ and a trivial choice of integration measure $g(q) = 1$. Such specific restrictions however, could prove to be incompatible with other properties that a particular system may be required to possess on physical or geometrical grounds.

Given the representation in (16), it is also possible to specify the properties[12] required of the wave functions $\psi(q)$ for the momentum operators $\hat{p}_\alpha$ to be self-adjoint. A straightforward analysis thus finds that these conditions are expressed by the surface integrals

$$\int_M d^nq\,\frac{\partial}{\partial q^\alpha}\left[\sqrt{g(q)}\;|< q|\psi >|^2\right] = \int_M d^nq\,\frac{\partial}{\partial q^\alpha}\left[\sqrt{g(q)}\,|\psi(q)|^2\right] = 0 \quad , \quad \alpha = 1,2,\ldots,n \quad . \tag{17}$$

Note how this set of conditions is independent of the choice of normalisation and of local phase of the position eigenbasis $|q >$, and thus indeed determines the domain of *states* $|\psi >$ for which the operators $\hat{p}_\alpha$ possess a self-adjoint representation on the considered space of states. Nevertheless, *given a choice of normalisation function* $g(q)$, the surface integrals (17) determine the necessary conditions defining the domain of the associated *wave functions* $\psi(q)$ for which the *differential operators* (16) are self-adjoint.

Another fundamental aspect of the momentum operators $\hat{p}_\alpha$ is that, being the generators of local translations on M, they induce through the representation (12) the parallel transport of states along curves in M in a manner consistent with the phase transformation induced by the flat U(1) bundle. In particular, when taken along a closed cycle starting and ending at a given point in M of coordinates q^α, the corresponding position eigenstate $|q >$ is thus transformed back into itself, *up to a phase factor given by the holonomy of the U(1) connection* $A_\alpha(q)$ *along that cycle*. Hence, it is only for a trivial U(1) bundle that this holonomy is always trivial, *i.e.* equal to unity, whatever the choice of cycle.

To make this remark more explicit, let us consider a basis of states which diagonalises all momentum operators $\hat{p}_\alpha$. Indeed, since all these hermitian operators commute with one another, they may be diagonalised together in terms of states $|p >$ spanning the whole space of states,

$$\hat{p}_\alpha = p_\alpha\,|p > \quad , \quad p_\alpha \,\epsilon\, \mathcal{D}(p) \quad . \tag{18}$$

[11] That of $\hat{q}^\alpha$ is of course given by $< q|\hat{q}^\alpha|\psi >= q^\alpha\,\psi(q)$.

[12] These are certainly necessary conditions, but the issue of whether they are also sufficient shall not be addressed specifically in the present paper, since this may be considered only on a case by case basis.

The range $\mathcal{D}(p)$ of spectral values p_α is left unspecified at this stage however, since it depends both on the topology of the base manifold M as well as on the considered space of states $|\psi>$ obeying the conditions (17). The other properties of the momentum eigenbasis $|p>$ left undetermined are their normalisation and phase. By analogy with the normalisation of the position eigenbasis $|q>$ in (4), the normalisation of the momentum eigenstates $|p>$ is parametrised according to[13]

$$< p|p' >= \frac{1}{\sqrt{h(p)}}\,\delta^{(n)}\left(p-p'\right) \quad , \tag{19}$$

where $h(p)$ is an arbitrary positive definite function defined over the domain $\mathcal{D}(p)$ of momentum eigenvalues p_α ($\alpha = 1, 2, \cdots, n$). Clearly, this choice still leaves the phase of the states $|p>$ as a free degree of freedom to be specified presently.

Note also that the above choice implies the spectral decomposition of the unit operator as

$$\mathbb{1} = \int_{\mathcal{D}(p)} d^n p\,\sqrt{h(p)}\,|p><p| \quad . \tag{20}$$

The quantities of direct relevance to the change of basis which is being considered, are the wave functions $<q|p>$. According to the configuration space representation (16) of the momentum operators $\hat{p}_\alpha$, and the fact that $|p>$ is a momentum eigenstate, these wave functions are determined by the following set of differential equations[14]

$$-i\hbar\frac{\partial}{\partial q^\alpha}\left[g^{1/4}(q)\ <q|p>\right] + A_\alpha(q)\,g^{1/4}(q)\ <q|p>= p_\alpha\,g^{1/4}(q)\ <q|p> \quad . \tag{21}$$

The construction of a solution to these equations proceeds as follows. Since they define a set of first order differential equations, they require one integration constant, namely the wave function $<q_0|p>$ associated to a specific point on M of coordinates q_0^α. Then, given any such specific point q_0 chosen arbitrarily, consider all other points in M of coordinates q^α, and for each of these points, an oriented path $P(q_0 \to q)$ running from q_0 to q. In addition, as functions of the end point q, this network of paths attached to a given q_0 must also define a *continuous* set. Given such data, the solution to (21) is of the form

$$g^{1/4}(q) <q|p>= \Omega\left[P(q_0\to q)\right]\,e^{\frac{i}{\hbar}(q-q_0)\cdot p}\,g^{1/4}(q_0) <q_0|p> \quad , \tag{22}$$

where the notation $(q\cdot p)$ stands of course for the sum $(q^\alpha p_\alpha)$, and where $\Omega\left[P(q_0\to q)\right]$ is the path ordered U(1) holonomy along the path $P(q_0 \to q)$,

$$\Omega\left[P(q_0\to q)\right] = Pe^{-\frac{i}{\hbar}\int_{P(q_0\to q)} dq^\alpha A_\alpha(q)} \quad . \tag{23}$$

However, the normalisation condition (19) then also requires that

$$\left|g^{1/4}(q_0) <q_0|p>\right|^2 = \frac{1}{(2\pi\hbar)^n}\,\frac{1}{\sqrt{h(p)}} \quad , \tag{24}$$

so that necessarily

$$g^{1/4}(q_0) <q_0|p>= \frac{e^{i\varphi(q_0,p)}}{(2\pi\hbar)^{n/2}}\,\frac{1}{h^{1/4}(p)} \quad , \tag{25}$$

[13] The notation used throughout applies when the momentum eigenvalues p_α form a continuous spectrum. In the case of a discrete spectrum, the relevant expressions have to be adapted appropriately, in an obvious manner.

[14] Note that these equations mean that the combination $e^{-\frac{i}{\hbar}q^\alpha p_\alpha} g^{1/4}(q) <q|p>$ is covariantly constant for the covariant derivative associated to the U(1) gauge transformations of states.

where $\varphi(q_0,p)$ is a specific but otherwise arbitrary real function.

Hence finally, the momentum eigenstate configuration space wave functions are given by[15]

$$< q|p >= \frac{e^{i\varphi(q_0,p)}}{(2\pi\hbar)^{n/2}}\,\frac{\Omega\left[P(q_0\rightarrow q)\right]}{g^{1/4}(q)h^{1/4}(p)}\,e^{\frac{i}{\hbar}(q-q_0)\cdot p}\quad , \tag{26}$$

generalising in a transparent manner the usual plane wave solutions of application to the trivial representation of the Heisenberg algebra with $A_\alpha(q)=0$ and with the choices $g(q)=1$ and $h(p)=1$. It thus appears that the arbitrary function $\varphi(q_0,p)$ is directly related to a phase convention for the momentum eigenbasis $|p>$. A specific choice could be such that,

$$e^{i\varphi(q_0,p)}\,e^{-\frac{i}{\hbar}q_0\cdot p}\,=\,1\quad , \tag{27}$$

thereby absorbing into the phase of the momentum eigenbasis $|p>$ the dependency of $< q|p >$ on the point q_0 through the phase function $\varphi(q_0,p)$. Such a restriction shall not be implemented here.

Momentum wave functions $< q|p >$ are thus constructed in terms of a base point q_0 and a continuous network of paths $P(q_0\rightarrow q)$ in M. The possible direct and rather trivial dependency on q_0 through $\varphi(q_0,p)$ has just been mentioned, while that on the path $P(q_0\rightarrow q)$ appears through the holonomy factor $\Omega\left[P(q_0\rightarrow q)\right]$. In the case of a trivial flat U(1) bundle $A_\alpha(q)$, this holonomy factor may be gauged away altogether through a phase redefinition of the position eigenbasis $|q>$ and an appropriate redefinition of the function $\varphi(q_0,p)$. For a non trivial flat bundle however, such gauging away is not possible globally over M, so that momentum wave functions $< q|p >$ carry with them generally such a holonomy phase factor dependent both on the path $P(q_0\rightarrow q)$ and the flat U(1) connection $A_\alpha(q)$.

These remarks also indicate how the functions $< q|p >$ depend on the choices of q_0 and the network of paths $P(q_0\rightarrow q)$. Changing these choices implies thus a change in the phase of the wave functions $< q|p >$, possibly through a redefinition both of the function $\varphi(q_0,p)$ and the holonomies $\Omega\left[P(q_0\rightarrow q)\right]$. In fact, since the network of paths is continuous, the change in holonomy would be at most only through a phase factor common to all wave functions $< q|p >$ and associated to the holonomy of $A_\alpha(q)$ along some homotopically non trivial closed cycle. Since the U(1) connection is flat, that latter holonomy in fact only depends on the homotopy class generator to which that cycle belongs, and is thus a constant factor independent of q_0 and p. Hence, any dependency on the choices of base point q_0 and network of paths $P(q_0\rightarrow q)$ may always be absorbed into a redefinition of the function $\varphi(q_0,p)$, namely into the phase of the momentum eigenbasis $|p>$, without any physical consequences.

These conclusions having been reached, it is now straightforward to show that the momentum operators $\hat{p}_\alpha$ do indeed generate translations in the configuration space manifold M, namely

$$Pe^{-\frac{i}{\hbar}\int_{P(q_0\rightarrow q)}dq^\alpha\,\hat{p}_\alpha}\;|q_0>=\frac{g^{1/4}(q)}{g^{1/4}(q_0)}\,\Omega\left[P(q_0\rightarrow q)\right]\,\Omega^{-1}\left[P(q_0\rightarrow q_0)\right]\,|q>\quad . \tag{28}$$

Here, $P(q_0\rightarrow q)$ is a path belonging to a continuous network of paths with q_0 as base point, while $P(q_0\rightarrow q_0)$ is the specific closed path of that network connecting q_0 to itself, which thus characterizes the homotopy class of the network of paths. Hence, in the limit that the point q is taken along a closed cycle $C(q_0\rightarrow q_0)$, the momentum operators $\hat{p}_\alpha$ indeed take the state $|q_0>$

[15]It may easily be checked that the normalisation condition (4) is then also satisfied. Moreover, note how the phase factor $e^{iq\cdot p/\hbar}$ constrains the spectral values of $\hat{p}_\alpha$, in a manner depending both on the topology of the configuration space manifold M (for example, whether it is bounded, periodic, or otherwise) as well as on the boundary conditions (17), thereby determining the domain $\mathcal{D}(p)$ of momentum eigenvalues.

back to itself, up to the holonomy $\Omega\left[C(q_0 \to q_0)\right]$ of the flat U(1) bundle along that cycle, as well as the U(1) mapping $\Omega^{-1}\left[P(q_0 \to q_0)\right]$ of the homotopy class of the network of paths used,

$$Pe^{-\frac{i}{\hbar}\int_{C(q_0 \to q_0)} dq^\alpha\,\hat{p}_\alpha}\ |q_0> = \Omega\left[C(q_0 \to q_0)\right]\ \Omega^{-1}\left[P(q_0 \to q_0)\right]\ |q_0> \quad . \tag{29}$$

When considered for all elements of the mapping class group $\pi_1(M)$, the phases $\Omega\left[C(q_0 \to q_0)\right]$ are precisely the functions which determine the embedding of the generators of the first homotopy group into the gauge group U(1), namely the representation of the Heisenberg algebra effectively being considered.

As a last application of the above developments solely based on the algebraic and topological properties of the Heisenberg algebra representation, let us consider the construction of a phase space path integral representation of the evolution operator of a quantised system,

$$U(t_f, t_i) = e^{-\frac{i}{\hbar}(t_f - t_i)\hat{H}} \quad . \tag{30}$$

Here, $\hat{H}$ is the quantum Hamiltonian of a given system, assumed to be self-adjoint on the considered representation space. We choose to discuss this point here rather than at the end of the next section, to emphasize the fact that, while the definition of the quantum Hamiltonian requires that the configuration manifold M be endowed with a geometrical structure, the construction of a path integral representation *per se* only requires the algebraic and topological considerations developed so far solely on basis of the Heisenberg algebra and assumptions A1 and A2 above.

Given the configuration space matrix elements $< q_f|U(t_f, t_i)|q_i >$ of the evolution operator, the steps leading to a phase space path integral representation of that quantity are well known (see for example Ref.[4]). In the present general setting, the main differences are in the normalisation factors $g(q)$ and $h(p)$ for position and momentum eigenstates, as well as in the phase factors related to the U(1) holonomies appearing in the matrix elements $< q|p >$ as constructed above. Except for these differences, the technical details are those of the usual development, leading in the present case to the expression

$$< q_f|U(t_f, t_i)|q_i > = \frac{\Omega\left[P(q_0 \to q_f)\right]\Omega^{-1}\left[P(q_0 \to q_i)\right]}{g^{1/4}(q_f)g^{1/4}(q_i)} \times$$

$$\times \lim_{N\to\infty} \int_M \prod_{i=1}^{N-1} d^n q_i \int_{\mathcal{D}(p)} \prod_{i=0}^{N-1} \frac{d^n p_i}{(2\pi\hbar)^n}\, e^{\frac{i}{\hbar}\sum_{i=0}^{N-1}\epsilon\left[\frac{q_{i+1}-q_i}{\epsilon}\cdot p_i - h_i\right]} \quad , \tag{31}$$

where,

$$\epsilon = \frac{t_f - t_i}{N} \quad , \quad h_i = \frac{< p_i|\hat{H}|q_i >}{< p_i|q_i >} = h_i^* \quad , \quad i = 0, 1, \cdots, N-1 \quad . \tag{32}$$

Note how any dependency on the normalisation factors $g(q)$ and $h(p)$ and the holonomies $\Omega\left[P(q_0 \to q)\right]$ has dropped from the phase space integration measure defining the discretized form of the path integral. The only dependency on these factors stems from the external states $|q_i>$ and $|q_f>$ in a consistent manner. As a matter of fact, apart from the overall holonomy factors $\Omega\left[P(q_0 \to q_i)\right]$ and $\Omega\left[P(q_0 \to q_f)\right]$, the only other dependency of the path integral on the flat U(1) connection $A_\alpha(q)$ on M, is implicit through the normalised matrix elements h_i of the quantum Hamiltonian operator.

It should be emphasized that the construct (31) provides an exact representation of the matrix element $< q_f|U(t_f, t_i)|q_i >$, whose discretized form is entirely and solely determined by the properties of the representation of the Heisenberg algebra being used. No ambiguity whatsoever arises for the discretized form of the phase space path integral, with in particular an integration measure which is completely specified, and is independent of the normalisation factors $g(q)$ and $h(p)$. This expression is thus valid under all circumstances, whether the configuration manifold is curved or flat, or whether it is parametrised by curvilinear coordinates or not. Note also that in those cases for which the integrations over the discretized conjugate momenta variables $p_{\alpha i}$ ($\alpha = 1, 2, \ldots, n; i = 0, 1, \ldots, N-1$) are feasable, the above path integral reduces to a discretized path integral over configuration space, with the appropriate integration measure factors for the coordinate variables q_i^α ($\alpha = 1, 2, \ldots, n; i = 1, 2, \ldots, N-1$) obtained without any ambiguity either, as well as a discretized expression for the Lagrangian appearing in the exponential factor of the integrand.

4 The Physical Setting: Geometry and Dynamics

Having understood how general configuration space representations of the Heisenberg algebra may be constructed over any arbitrary differentiable manifold M, in a manner involving only algebraic and topological considerations, let us apply these results to the dynamical description of a given physical system. Quite generally, the description of such dynamics requires first the specification of a geometry on the configuration space M of the system, in order to define a variational principle in terms of a Lagrange function. Typically, such a Lagrange function is of the form

$$L = \frac{1}{2} m g_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta \, - \, V(q) \quad , \tag{33}$$

the notation being self-explanatory. In particular, the functions $g_{\alpha\beta}(q)$ define the components of a metric on the configuration space M of the system, expressed in the coordinate system specified by the variables q^α. Note that such a description encompasses both general Riemannian manifolds of non vanishing scalar curvature, as well as flat spaces parametrised by curvilinear coordinates, such as for example the n dimensional euclidean space parametrised by spherical coordinates[16]. In this sense, one may say that dynamics requires geometry, while quantisation requires algebra and topology. The purpose of this section is to show how quantum dynamics thus results from the marriage of algebra, topology and geometry.

Given the Lagrange function (33), the canonical Hamiltonian analysis of such a system is straightforward, leading to momenta p_α canonically conjugate to the coordinates q^α—whose Poisson brackets $\{q^\alpha, p_\beta\} = \delta^\alpha_\beta$ are thus canonical—and a time evolution generated by the classical Hamiltonian

$$H_0 = \frac{1}{2m} \, p_\alpha \, g^{\alpha\beta}(q) p_\beta \, + \, V(q) \quad , \tag{34}$$

$g^{\alpha\beta}(q)$ being the inverse of the metric $g_{\alpha\beta}(q)$.

Canonical quantisation of such a system is then defined by the Heisenberg algebra (1) associated to the configuration space M, thereby leading to a space of quantum states which provides a representation of that algebra, and thus belongs to one of the constructions described in Sects.2 and 3 involving a flat U(1) bundle and its topological characterization in terms of the first homotopy group $\pi_1(M)$. Which of these representations is to be used for a given physical system depends on the observed physical properties of that system[17].

Moreover, the dynamics of the quantised system derives from its quantum Hamiltonian $\hat{H}$ which, through the correspondence principle, should be in correspondence with the classical Hamiltonian (34). However, this principle does not specify uniquely the *quantum* Hamiltonian, since two such operators differing only by terms of order $\hbar$ (or $\hbar^2$ for a time reversal invariant system) correspond to identical dynamics at the classical level. In fact, in the same way as for

[16] Indeed, the representation theory of the Heisenberg algebra in coordinate systems other than cartesian is usually problematic[6], and the approach of the previous two sections should precisely provide the adequate framework to address these issues, beginning with curvilinear coordinates on an euclidean manifold.

[17] In the same way that the choice of representations of the $su(2)$ algebra to be used for a space rotationally invariant quantum system for example, depends on the physical spin content of that system.

the Heisenberg algebra representation, the choice of a quantum Hamiltonian in correspondence with a classical one is also a matter of physics, namely of the physical properties of the dynamics of the (quantum) system being considered, the only other restriction of application being that the quantum Hamiltonian defines a self-adjoint operator on the space of states, in order to ensure a unitary time evolution.

This freedom in the choice of quantum dynamics is demonstrated by the following two variable parametrisation of quantum Hamiltonians $\hat{H}_{\mu,\nu}$, which are in direct correspondence with the classical Hamiltonian H_0 in (34),

$$\hat{H}_{\mu,\nu}=\frac{1}{2m}\,\hat{g}^{-1/4+\mu-i\nu}(\hat{q})\;\hat{p}_\alpha\;\hat{g}^{1/2-2\mu}(\hat{q})\;\hat{g}^{\alpha\beta}(\hat{q})\;\hat{p}_\beta\;\hat{g}^{-1/4+\mu+i\nu}(\hat{q})\;+\;\hat{V}(\hat{q})\quad, \tag{35}$$

where μ and ν are two arbitrary real variables, while $\hat{q}^\alpha$ and $\hat{p}_\alpha$ are the self-adjoint operators whose representations where developed in Sects.2 and 3. By construction, the latter properties of $\hat{q}^\alpha$ and $\hat{p}_\alpha$ should ensure that $\hat{H}_{\mu,\nu}$ is indeed a self-adjoint operator on the considered representation space of the Heisenberg algebra. Hence, $\hat{H}_{\mu,\nu}$ is *a priori* a perfectly acceptable time evolution operator for the quantised system, and only physical properties to be observed may determine whether the association of $\hat{H}_{\mu,\nu}$ for specific values of μ and ν to a given physical system is indeed appropriate.

Note that the existence of such a two variable parametrisation of self-adjoint quantum Hamiltonians in correspondence with a common classical one, is reminiscent of von Neumann's theory of self-adjoint extensions of differential operators[2]. In fact, there may exist some correspondence between the parameters μ and ν introduced here, and the von Neumann deficiency indices characterizing the self-adjoint extensions of the quadratic differential operator associated to the Hamiltonian $\hat{H}_{\mu=0,\nu=0}$ as well as to the trivial configuration space representation of the Heisenberg algebra whose U(1) connection $A_\alpha(q)$ vanishes identically. Being beyond the scope of the present paper, this issue shall not be pursued here.

As a matter of fact, the choice $(\mu=0,\nu=0)$ corresponds to the usual scalar Laplacian operator over the configuration manifold, which indeed provides for the canonical choice of quantum Hamiltonian. In the configuration space representation, its expression is

$$\hat{H}_{0,0}\quad:\quad\frac{-\hbar^2}{2m}\frac{1}{\sqrt{g(q)}}\left[\partial_\alpha+\frac{i}{\hbar}A_\alpha(q)\right]\sqrt{g(q)}\,g^{\alpha\beta}(q)\,\left[\partial_\beta+\frac{i}{\hbar}A_\beta(q)\right]\,+\,V(q)\quad, \tag{36}$$

where the ordinary derivatives $\partial_\alpha\equiv\partial/\partial q^\alpha$ are replaced by U(1) covariant derivatives. Thus for example, in the case of a system whose configuration space is the n dimensional euclidean space, and for which it is established on physics grounds that its quantum dynamics is governed by the scalar Laplacian operator in cartesian coordinates, it is the Hamiltonian $\hat{H}_{0,0}$ which is to be used for a description of the same quantum system in any curvilinear parametrisation of its configuration space.

On the other hand, the choice $(\mu=\frac{1}{4},\nu=0)$ leads to a quantum Hamiltonian $\hat{H}_{1/4,0}$ in direct naive correspondence with the classical Hamiltonian (34),

$$\hat{H}_{\frac{1}{4},0}=\frac{1}{2m}\hat{p}_\alpha\,\hat{g}^{\alpha\beta}(\hat{q})\,\hat{p}_\beta\,+\,\hat{V}(\hat{q})\quad. \tag{37}$$

However, this operator does not produce the U(1) invariant scalar Laplacian operator on the configuration manifold, and thus in general determines a quantum dynamics different from that governed by $\hat{H}_{0,0}$, even when the configuration space M and its geometry $g_{\alpha\beta}(q)$ are identical, as well as the representation of the Heisenberg algebra which is being used, and which is characterized through the topological class of the flat U(1) connection $A_\alpha(q)$.

In fact, all the operators $\hat{H}_{\mu,\nu}$ differ by terms of order $\hbar^2$, so called a "quantum correction potential". An explicit calculation finds

$$\hat{H}_{\mu,\nu}-\hat{H}_{0,0}=\hat{\Delta}_{\mu,\nu}\quad, \tag{38}$$

with the function $\Delta_{\mu,\nu}$ given by

$$\Delta_{\mu,\nu} = \frac{\hbar^2}{2m}\, g^{\mu-1/2}\, \partial_\alpha \left[g^{1/2} g^{\alpha\beta}\, \partial_\beta\, g^{-\mu} \right] \;+\; \frac{\hbar^2\, \nu^2}{2m}\, g^{-2}\, g^{\alpha\beta}\, \partial_\alpha\, g\, \partial_\beta\, g\, + \\ + \frac{\hbar\, \nu}{2m} \left[\left(g^{-1} g^{\alpha\beta}\, \partial_\beta\, g \right) p_\alpha \,+\, p_\alpha \left(g^{-1} g^{\alpha\beta}\, \partial_\beta\, g \right) \right] \quad , \tag{39}$$

where in this last expression, care has to be exercised in not commuting variables which at the quantum level correspond to non commuting operators. In general, unless this quantity $\Delta_{\mu,\nu}$ vanishes identically, the physics implied by the Hamiltonians $\hat{H}_{\mu,\nu}$ and $\hat{H}_{0,0}$ are clearly different.

The point made above concerning the invariance under changes of coordinates of the quantum physics properties of a system whose n dimensional euclidean space is parametrised by curvilinear coordinates, also brings us to an important property of the representations of the Heisenberg algebra constructed in Sects.2 and 3. As was already indicated there, when the configuration space manifold M is endowed with a Riemannian geometrical structure, which is now required by the dynamics, the canonical choice for the normalisation function $g(q)$ of the position eigenbasis $|q>$ is the determinant $g(q) = \det\, g_{\alpha\beta}(q)$ of the Riemannian metric. What makes such a choice then in fact compulsory are the relations (6) and (7). Indeed, these identities establish that the configuration space wave functions $\psi(q) =< q|\psi >$ of all quantum states $|\psi >$, as well as the inner product of these states in terms of their wave functions, are then manifestly covariant under changes in the coordinate parametrisation of the configuration space manifold, a fact which stems from the diffeomorphic invariant property of the integration measure $d^n q \sqrt{\det\, g_{\alpha\beta}(q)}$ over the Riemannian manifold M. Hence, not only do the representations constructed in Sects.2 and 3 provide the most general possible representations of the Heisenberg algebra over an arbitrary manifold M, but also, and as importantly, they define diffeomorphic covariant representations in the case of a Riemannian manifold. In other words, given a physical system, as well as the associated representation of the Heisenberg algebra over its configuration space, and finally the associated quantum Hamiltonian $\hat{H}_{\mu,\nu}$, the description of the physical properties and time evolution of the system which then ensue, is entirely independent of the chosen coordinate parametrisation of its configuration space.

One last issue to be addressed more specifically is the self-adjoint property required of the quantum Hamiltonian. Formally, given the expression (35) which defines $\hat{H}_{\mu,\nu}$, these operators appear to meet that condition whatever the values for the parameters μ and ν, since the operators $\hat{q}^\alpha$ and $\hat{p}_\alpha$ are assumed to be self-adjoint. Nevertheless, it proves useful to translate that requirement also in terms of the wave functions $\psi(q,t) =< q|\psi, t >$, which are already constrained to obey the surface term conditions (17). One consequence of the unitarity of the evolution operator generated by a given quantum Hamiltonian $\hat{H}_{\mu,\nu}$, is the existence of a conserved probability current $J_\alpha(q,t)$ in the configuration space wave function representation, such that

$$\frac{\partial}{\partial t}\rho(q,t) \,+\, \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^\alpha}\sqrt{g}g^{\alpha\beta}\, J_\beta(q,t) \,=\, 0 \quad , \tag{40}$$

with the probability density

$$\rho(q,t) = |< q|\psi, t >|^2 \quad . \tag{41}$$

Hence, configuration space wave functions $\psi(q,t)$ are constrained to satisfy both the conditions (17) necessary for the self-adjoint property of the momentum operators $\hat{p}_\alpha$, as well as the probability conservation constraint (40) at all points of the configuration space Riemannian manifold M. Note that whatever the specific relationship between $J_\alpha(q,t)$ and the wave function $\psi(q,t)$, these two classes of restrictions involve both the wave function itself and its first order derivatives with respect to the coordinates q^α.

A specific relationship between $\psi(q,t)$ and $J_\alpha(q,t)$ follows only a choice for the quantum Hamiltonian. Given the operators $\hat{H}_{\mu,\nu}$ in (35), the associated probability conservation equation reads as in (40), with a current density $J_\alpha(q)$ which defines the U(1) covariant generalisation of the usual expression and includes a dependency on the parameters μ and ν,

$$J_\alpha = -\frac{i\hbar}{2m}\, g^{-2\mu}\left[\left(g^{\mu+i\nu}\psi\right)^*\left(\partial_\alpha + \frac{i}{\hbar}A_\alpha\right)\left(g^{\mu+i\nu}\psi\right) - \left(\left(\partial_\alpha + \frac{i}{\hbar}A_\alpha\right)\left(g^{\mu+i\nu}\psi\right)\right)^*\left(g^{\mu+i\nu}\psi\right)\right] \quad . \tag{42}$$

Note that here as well, the issue—which is beyond the scope of the present paper—of the possible relation between the variables μ and ν and the von Neumann deficiency indices for the self-adjoint extensions of $\hat{H}_{0,0}$ when $A_\alpha(q) = 0$, arises again but in another disguise. Indeed, in the case of a configuration space with boundaries, the conservation of probability at the boundaries requires that the current $J_\alpha(q)$ vanishes at those points, thereby implying specific relations between the wave function $\psi(q,t)$ and its first order derivatives $\partial_\alpha\psi(q,t)$ at the boundaries. Typically, such relations precisely arise also in the discussion of self-adjoint extensions of quadratic differential operators, and are indeed in direct correspondence with the von Neumann deficiency indices characterizing such self-adjoint extensions.

5 The Free Particle in Two Euclidean Dimensions

As a simple illustration of the previous general discussion, let us consider the case of a non relativistic particle of mass m free to propagate in a two dimensional euclidean space. Such a space being simply connected, there thus exists, up to unitary transformations, a single representation only of the associated Heisenberg algebra, namely the trivial one whose U(1) flat connection vanishes identically, $A_\alpha(q) = 0$, while the normalisation factor $g(q)$ is determined by the flat euclidean geometry of the configuration space. However, rather than working in cartesian coordinates, a polar parametrisation of the manifold will be used to demonstrate that the general analysis developed previously is perfectly adequate to consider canonical quantisation in curvilinear coordinates.

With the choice of parameters $(q^1, q^2) = (q^r, q^\theta) = (r, \theta)$ defining polar coordinates in the plane, the corresponding metric tensor is given by

$$ds^2 = g_{\alpha\beta}(q)dq^\alpha dq^\beta = dr^2 + r^2 d\theta^2 \quad , \tag{43}$$

with in particular $g(q) = r^2$ specifying the normalisation of the position eigenstates $|q>$ through (4). Consequently, according to (16), the associated configuration space representation of the momentum operators $\hat{p}_r$ and $\hat{p}_\theta$ is

$$\hat{p}_r : \quad -\frac{i\hbar}{\sqrt{r}}\,\partial_r\sqrt{r} \quad , \quad \hat{p}_\theta : \quad -i\hbar\,\partial_\theta \quad . \tag{44}$$

Given these operators, let us turn to the choice of Hamiltonian determining the quantum dynamics of the system, for which we shall consider the general abstract operator $\hat{H}_{\mu,\nu}$ defined in (35) with a vanishing potential $\hat{V}(\hat{q})$. In the configuration space representation being used, this expression reduces to the following differential operator,

$$\hat{H}_{\mu,\nu} : \quad -\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{(1+4i\nu)}{r}\partial_r - \frac{4(\mu^2+\nu^2)}{r^2} + \frac{1}{r^2}\partial_\theta^2\right] \quad . \tag{45}$$

Determining the energy eigenstates of this operator is a straightforward exercise. The eigenvalue spectrum is given by all real and positive values of the energy, $E \geq 0$, as well as all positive, null and negative integer values of the angular momentum, ℓ, with the associated configuration space eigenstate wave functions determined by,

$$\psi_{E,\ell}(r,\theta) = \left(\frac{m}{2\pi\hbar^2}\right)^{1/2} e^{i\ell\theta}\left(r\sqrt{\frac{2mE}{\hbar^2}}\right)^{-2i\nu} J_{|\alpha|}\left(r\sqrt{\frac{2mE}{\hbar^2}}\right) \quad , \tag{46}$$

having introduced

$$|\alpha| = \sqrt{\ell^2 + 4\mu^2} \quad , \tag{47}$$

and where $J_{|\alpha|}(x)$ is the Bessel function of the first kind of order $|\alpha|$. These eigenstate wave functions are normalised such that,

$$< \psi_{E,\ell} | \psi_{E',\ell'} >= \delta_{\ell\ell'}\, \delta(E - E') \quad , \tag{48}$$

with the inner product defined by (7) as well as $g(r, \theta) = r^2$.

The Bessel functions of the second kind $N_{|\alpha|}(r\sqrt{2mE/\hbar^2})$ do also define solutions to the Schrödinger equation associated to the above differential operator. However, such configurations are excluded on the grounds that the corresponding probability density current $J_\alpha(q)$ constructed in (42) should remain finite throughout the entire two dimensional plane, and in particular at the origin $r = 0$.

It is clear that in the limit where $(\mu, \nu) = (0, 0)$, the solutions in (46) reduce to the eigenfunctions of the scalar Laplacian operator on the plane in polar coordinates. One among other ways to illustrate this point is to consider the usual normalised plane wave eigenfunctions of the scalar Laplacian operator expressed in cartesian coordinates, namely

$$\frac{1}{2\pi\hbar}\, e^{\frac{i}{\hbar}(xp_x + yp_y)} \quad . \tag{49}$$

Introducing the parametrisation,

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad ; \quad p_x = p\cos\varphi \quad , \quad p_y = p\sin\varphi \quad , \quad p = \sqrt{2mE} \geq 0 \quad , \tag{50}$$

these plane wave solutions are also expressed as

$$\frac{1}{2\pi\hbar}\, e^{\frac{i}{\hbar}(xp_x + yp_y)} = \sum_{\ell=-\infty}^{\infty} \frac{1}{\sqrt{2\pi m}} \left(i^{|\ell|} e^{-i\ell\varphi} \right) \left(\frac{m}{2\pi\hbar^2} \right)^{1/2} e^{i\ell\theta}\, J_{|\ell|}\left(r\sqrt{\frac{2mE}{\hbar^2}} \right) \quad . \tag{51}$$

In other words, the usual plane wave solutions associated to cartesian coordinates and the solutions constructed in (46) associated to polar coordinates, do indeed span the same space of quantum states in the case $(\mu, \nu) = (0, 0)$ which corresponds to the scalar Laplacian on the plane as the choice for quantum Hamiltonian.

Given the above spectrum of states for the Hamiltonian $\hat{H}_{\mu,\nu}$, it is also possible to determine the time evolution operator of the system. The evaluation of this propagator in configuration space, namely that of its position matrix elements $< r_f, \theta_f | e^{-i\Delta t \hat{H}_{\mu,\nu}/\hbar} | r_i, \theta_i >$, may proceed either from the explicit knowledge of the eigenstates of that operator, or from the calculation of the phase space path integral representation in (31). The former approach immediately leads to the expression[18]

$$< r_f, \theta_f | e^{-i\Delta t \hat{H}_{\mu,\nu}/\hbar} | r_i, \theta_i >=$$
$$= \frac{1}{2\pi\hbar^2} \left(\frac{r_i}{r_f} \right)^{2i\nu} \sum_{\ell=-\infty}^{\infty} e^{i\ell(\theta_f - \theta_i)} \int_0^\infty dp\, p\, e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}\, J_{|\alpha|}\left(\frac{r_f p}{\hbar} \right) J_{|\alpha|}\left(\frac{r_i p}{\hbar} \right) \quad . \tag{52}$$

Again in the particular case $(\mu, \nu) = (0, 0)$, the final summation over the angular momentum ℓ may explicitely be done using one of the addition theorems for Bessel functions. One then finds,

$$< r_f, \theta_f | e^{-i\Delta t \hat{H}_{0,0}/\hbar} | r_i, \theta_i >= \frac{m}{2i\pi\hbar\Delta t}\, e^{-\frac{m}{2i\hbar\Delta t}\left(r_f^2 + r_i^2 - 2r_f r_i \cos(\theta_f - \theta_i) \right)} = \frac{m}{2i\pi\hbar\Delta t}\, e^{-\frac{m}{2i\hbar\Delta t}\left(\vec{x}_f - \vec{x}_i \right)^2} \quad , \tag{53}$$

[18] The final integration over the momentum p may be accomplished in terms of a series involving hypergeometric functions, which is not very illuminating.

an expression which is indeed recognized as that of the propagator for a non relativistic free particle in two dimensions, whose Hamiltonian is simply the canonical choice $\hat{H} = \hat{\vec{p}}^2/(2m)$ associated to the scalar Laplacian on the euclidean plane.

A similar analysis starting from the evaluation of the discretized phase space path integral (31) may be developed, leading to precisely identical conclusions. However, since this calculation will be discussed in the next section for a system which includes the free particle as a limiting case, no further details are given here.

Nevertheless, let us emphasize that through the general approach developed in the previous sections, a genuine canonical quantisation of the Heisenberg algebra associated to curvilinear coordinates and leading to physically correct results is indeed possible, as the simple illustration of the present section has demonstrated.

6 The Spherical Harmonic Oscillator in a Punctured Plane

As a second example, let us consider the two dimensional spherical harmonic oscillator of mass m and angular frequency ω oscillating in a punctured plane of which the origin $r = 0$ has been removed. The harmonic potential $V(r) = m\omega^2 r^2/2$ thus serves the purpose of an infrared regularisation of the solutions to the Schrödinger equation through the confinement of the particle within the potential well, while the removal of the origin of the plane induces a non trivial topology of the configuration space of this system.

Indeed, due to the non trivial first homotopy group $\pi_1(M) = \mathbb{Z}$ in this case, the Heisenberg algebra admits an infinity of unitarily inequivalent representations, labelled by the holonomy of a flat U(1) gauge field $A_\alpha(q)$ around the origin $r = 0$. Up to U(1) gauge transformations, such flat U(1) bundles may all be characterized by the differential 1-form

$$dq^\alpha \, A_\alpha(q) = \hbar\lambda d\theta \quad , \tag{54}$$

where λ is an arbitrary real parameter which thus labels the different representations of the Heisenberg algebra. In particular, the choice $\lambda = 0$ reproduces the trivial representation which appeared in the previous section, and which is thus equivalent to having the two dimensional plane without the puncture at $r = 0$.

Consequently, working still in polar coordinates, the configuration space representations of the momentum operators $\hat{p}_r$ and $\hat{p}_\theta$ are modified as follows,

$$\hat{p}_r : \quad -\frac{i\hbar}{\sqrt{r}}\,\partial_r\,\sqrt{r} \quad , \quad \hat{p}_\theta : \quad -i\hbar\,(\partial_\theta + i\lambda) \quad . \tag{55}$$

Choosing the work with the general quantum Hamiltonian $\hat{H}_{\mu\nu}$ defined in (35), the associated differential operator then reads

$$\hat{H}_{\mu,\nu} : \quad -\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{(1+4i\nu)}{r}\partial_r - \frac{4(\mu^2+\nu^2)}{r^2} + \frac{1}{r^2}\,(\partial_\theta + i\lambda)^2\right] + \frac{1}{2}m\omega^2 r^2 \quad . \tag{56}$$

Here again, it is rather straightforward to determine the eigenspectrum of this differential operator, thereby solving the associated Schrödinger equation. One then finds that the energy spectrum of the system is discrete and given by

$$E_{m,\ell} = \hbar\omega\Big[2m + 1 + |\alpha|\Big] \quad , \quad m = 0,1,2,\ldots \quad , \quad \ell = 0,\pm1,\pm2,\ldots \quad , \tag{57}$$

with the quantity $|\alpha|$ defined as

$$|\alpha| = \sqrt{(\ell+\lambda)^2 + 4\mu^2} \quad , \tag{58}$$

ℓ being the angular momentum of the states, as previously. The corresponding wave functions are[19]

$$\psi_{m,\ell}(r,\theta)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\sqrt{\frac{m!}{\Gamma(|\alpha|+m+1)}}\,e^{i\ell\theta}\,u^{-2i\nu}\,u^{|\alpha|}\,e^{-\frac{1}{2}u^2}\,L_m^{|\alpha|}(u^2)\quad, \tag{59}$$

with

$$u=r\sqrt{\frac{m\omega}{\hbar}}\quad, \tag{60}$$

while the $L_m^{|\alpha|}(x)$ are the usual Laguerre polynomials. These states are normalised according to

$$<\psi_{m,\ell}|\psi_{m',\ell'}>=\delta_{mm'}\delta_{\ell\ell'}\quad. \tag{61}$$

Note how the presence of the non trivial U(1) connection parametrised by λ simply shifts the value of the angular momentum ℓ in the energy spectrum, thereby leading to a periodic spectral flow of the energy eigenvalues. In particular, the choice $(\mu,\nu)=(0,0)$ reproduces the well known spherical harmonic oscillator spectrum associated to the choice of the scalar Laplacian on euclidean space in the case of the trivial Heisenberg representation, namely when $\lambda=0$.

Having diagonalised the quantum Hamiltonian of the system, it becomes possible to compute its propagator in configuration space, in polar coordinates. Using the complete set of energy eigenstates constructed above, a straightforward analysis of the relevant matrix elements easily finds

$$<r_f,\theta_f|e^{-i\Delta t\hat{H}_{\mu,\nu}/\hbar}|r_i,\theta_i>=$$

$$=\frac{m\omega}{2i\pi\hbar\sin\omega\Delta t}\left(\frac{u_i}{u_f}\right)^{2i\nu}\,e^{\frac{i}{2}\frac{\cos\omega\Delta t}{\sin\omega\Delta t}\left(u_f^2+u_i^2\right)}\,\sum_{\ell=-\infty}^{\infty}\,e^{-i\frac{\pi}{2}|\alpha|}\,e^{i\ell(\theta_f-\theta_i)}\,J_{|\alpha|}\left(\frac{u_f u_i}{\sin\omega\Delta t}\right)\quad. \tag{62}$$

In particular, in the case of the trivial representation of the Heisenberg algebra, $\lambda=0$, as well as the canonical choice of Hamiltonian $(\mu,\nu)=(0,0)$, it is possible to show that this expression reduces to the usual and correct one for the propagator, namely,

$$<r_f,\theta_f|e^{-i\Delta t\hat{H}_{0,0}/\hbar}|r_i,\theta_i>_{|\lambda=0}=\frac{m\omega}{2i\pi\hbar\sin\omega\Delta t}\,e^{\frac{im\omega}{2\hbar\sin\omega\Delta t}\left[\cos\omega\Delta t(r_f^2+r_i^2)-2r_fr_i\cos(\theta_f-\theta_i)\right]}=$$

$$=\frac{m\omega}{2i\pi\hbar\sin\omega\Delta t}\,e^{\frac{im\omega}{2\hbar\sin\omega\Delta t}\left[\cos\omega\Delta t(\vec{x}_f^2+\vec{x}_i^2)-2\vec{x}_f\cdot\vec{x}_i)\right]}\quad. \tag{63}$$

The same expression as in (62) may be derived considering the discretized phase space path integral given in (31). The fact that both calculations lead to precisely the same result, whatever the values for μ, ν and λ, shows that the approach to the canonical quantisation of the Heisenberg algebra advocated in general terms in this paper is sound and physically consistent in the case of an arbitrary system of curvilinear coordinates on its configuration space. In particular, it is only with the specific choice of integration measure implicit in the discretized form of the path integral (31) that the same expression as the one given in (62) may be derived for the propagator.

More explicitely, the evaluation of the path integral (31) proceeds as follows. A careful analysis is required first of the abstract quantum Hamiltonian $\hat{H}_{\mu,\nu}$ in (35), in order to determine the normalised matrix elements,

$$h_i=\frac{<p_i,\ell_i|\hat{H}_{\mu,\nu}|r_i,\theta_i>}{<p_i,\ell_i|r_i,\theta_i>}\quad,\quad i=0,1,\cdots,N-1\quad. \tag{64}$$

[19]There exists another class of solutions to the second order differential Schrödinger equation, but those solutions are excluded on the requirement of normalisable energy eigenstate wave functions.

Here, p_i stands for the eigenvalue of the radial momentum operator $\hat{p}_r$, while the integer ℓ_i labels the angular momentum value which determines the eigenvalue of the angular momentum operator $\hat{p}_\theta$ as $\hbar(\ell_i+\lambda)$. A straightforward calculation then finds

$$h_i=\frac{1}{2m}\left[p_i^2+4\hbar\nu\,\frac{p_i}{r_i}+\frac{\hbar^2}{r_i^2}\left((\ell_i+\lambda)^2-\frac{1}{4}+4(\mu^2+\nu^2)-2i\nu\right)\right]+\frac{1}{2}m\omega^2r_i^2\quad . \tag{65}$$

Hence, in the present case, the discretised phase space path integral representation (31) of the propagator reads[20]

$$<r_f,\theta_f|e^{-i\Delta t\hat{H}_{\mu,\nu}/\hbar}|r_i,\theta_i>=\frac{\Omega[P(q_0\to q_f)]\Omega^{-1}[P(q_0\to q_i)]}{(r_fr_i)^{1/2}}\times$$

$$\times\lim_{N\to\infty}\int_0^\infty\prod_{i=1}^{N-1}dr_i\int_0^{2\pi}\prod_{i=1}^{N-1}d\theta_i\int_{-\infty}^{+\infty}\prod_{i=0}^{N-1}\frac{dp_i}{2\pi\hbar}\prod_{i=0}^{N-1}\left(\sum_{l_i=-\infty}^{+\infty}\frac{1}{2\pi}\right)\times$$

$$\times e^{\frac{i}{\hbar}\sum_{i=0}^{N-1}[(r_{i+1}-r_i)p_i+\hbar(\theta_{i+1}-\theta_i)(\ell_i+\lambda)-\epsilon h_i]}\quad . \tag{66}$$

The integrations and summations over the variables θ_i, p_i and ℓ_i are immediate, leading to the expression,

$$<r_f,\theta_f|e^{-i\Delta t\hat{H}_{\mu,\nu}/\hbar}|r_i,\theta_i>=\frac{\Omega[P(q_0\to q_f)]\Omega^{-1}[P(q_0\to q_i)]}{2\pi(r_fr_i)^{1/2}}\times$$

$$\times\sum_{\ell=-\infty}^{+\infty}e^{i\ell(\theta_f-\theta_i)}\lim_{N\to\infty}\left(\frac{m}{2i\pi\epsilon\hbar}\right)^{N/2}\int_0^\infty\prod_{i=1}^{N-1}dr_i\ \times$$

$$\times\prod_{i=0}^{N-1}e^{\frac{i}{\hbar}\left[\frac{m}{2\epsilon}(r_{i+1}^2+r_i^2)-\frac{1}{2}\epsilon m\omega^2r_i^2-\frac{\epsilon\hbar^2}{2m}\frac{1}{r_i^2}(|\alpha|^2-\frac{1}{4}-2i\nu)\right]}e^{-\frac{i}{\hbar}\frac{m}{\epsilon}r_ir_{i+1}}e^{-2i\nu(\frac{r_{i+1}}{r_i}-1)}\quad , \tag{67}$$

where the parameter $|\alpha|$ is already defined in (58).

The evaluation of the remaining integrals over the variables r_i ($i=1,2,\dots,N-1$) now proceeds by using the techniques developed in[21] Ref.[7]. After some work, one then finds exactly,

$$<r_f,\theta_f|e^{-i\Delta t\hat{H}_{\mu,\nu}/\hbar}|r_i,\theta_i>=\frac{m\omega}{2i\pi\hbar\sin\omega\Delta t}\left(\frac{r_i}{r_f}\right)^{2i\nu}\Omega[P(q_0\to q_f)]\Omega^{-1}[P(q_0\to q_i)]\times$$

$$\times e^{\frac{im\omega}{2\hbar}\frac{\cos\omega\Delta t}{\sin\omega\Delta t}(r_f^2+r_i^2)}\sum_{\ell=-\infty}^{+\infty}e^{-i\frac{\pi}{2}|\alpha|}\,e^{i(\ell+\lambda)(\theta_f-\theta_i)}\,J_{|\alpha|}\left(\frac{m\omega}{\hbar}\frac{r_fr_i}{\sin\omega\Delta t}\right)\quad . \tag{68}$$

Only the evaluation of the U(1) holonomies associated to the external states is still required, and should cancel the apparent lack of single-valuedness in the angular dependency on

[20] Contrary to the notations used in (31), the spectrum of the angular momentum operator $\hat{p}_\theta$ being discrete, with eigenvalues $\hbar(\ell+\lambda)$, the expression in (31) has to be adapted appropriately in terms of summations rather than integrations over the integers ℓ_i.

[21] Incidentally, note that the first reference in [7] considered the problem of determining the correct integration measure in a discretized form of the phase space path integral in polar coordinates starting from the knowledge of the correct propagator in cartesian coordinates, associated to the scalar Laplacian operator in the euclidean plane and the trivial representation of the Heisenberg algebra. Here, we recover in that specific instance the same conclusions of course, but rather by working from our general analysis of diffeomorphic covariant representations of the Heisenberg algebra on arbitrary manifolds, and including the possibility of non trivial U(1) holonomies as well as the general class of quantum Hamiltonians $\hat{H}_{\mu,\nu}$.

$(\theta_f - \theta_i)$ which may seem to occur for non integer values of the U(1) holonomy λ due to the factors $e^{i(\ell+\lambda)(\theta_f-\theta_i)}$ in this expression. Given the choice (54), it should be clear that we have

$$\Omega[P(q_0 \to q_f)]\Omega^{-1}[P(q_0 \to q_i)] = e^{-i\lambda(\theta_f-\theta_i)} \quad , \tag{69}$$

so that finally the result (68) does indeed reproduce *exactly*, including its absolute normalisation and phase factors, the propagator of the system evaluated in (62) directly on basis of the diagonalisation of the quantum Hamiltonian $\hat{H}_{\mu,\nu}$.

It should be emphasized that this conclusion is by no means trivial, since the two ways to computing this propagator are entirely different. This thus demonstrates once again the soundness of the general construction of configuration space representations of the Heisenberg algebra on arbitrary manifolds advocated in this paper. Note also that the presence of the non trivial holonomy factors $\Omega[P(q_0 \to q)]$ in the path integral representation (31) in the case of non trivial representations of the Heisenberg algebra, is essential to obtain the correct final expression, and to render the propagator indeed single-valued in multiply-connected configuration space variables such as the angle θ in the example of this section.

Before leaving this system, we should call attention to the following issue, related to the previously pointed out spectral flow of the energy spectrum as a function of λ. As is well known, in the case of the trivial representation of the Heisenberg algebra, $\lambda = 0$, as well as the choice of the scalar Laplacian operator as quantum Hamiltonian, $(\mu, \nu) = (0, 0)$, the system may easily be solved algebraically in terms of either cartesian or helicity-like creation and annihilation operators, the latter choice being related to polar coordinates and thus particularly appropriate for a system with circular symmetry[8]. In particular, the ground state Fock vacuum is annihilated by the annihilation operators of both helicities. However, still with the choice $(\mu, \nu) = (0, 0)$, as soon as $\lambda \neq 0$, namely for non trivial representations of the Heisenberg algebra, a Fock space construction of the spectrum of the quantised system still seems possible—at least the quantum Hamiltonian $\hat{H}_{0,0}$ factorizes in terms of the helicity creation and annihilation operators—, but then the vacuum state is at best annihilated by only one of the helicity annihilation operators, whose helicity depends on the sign of λ. Nevertheless, the energy spectrum flows with the values of λ and with an integer periodicity of unity. The question which this situation thus raises is whether a Fock space construction of the quantum spectrum is possible when $\lambda \neq 0$ (and also possibly when $(\mu, \nu) \neq (0, 0)$), and how such an approach would explain the spectral flow not only of the energy spectrum but also of its energy eigenstates, which define a basis of the space of states. Presumably, some type of Bogoliubov or coherent state transformation is required, but we shall leave this issue beyond the scope of this paper, whose primary motivation is the presentation of the construction of representations of the Heisenberg algebra on arbitrary manifolds.

7 Conclusions

The considerations developed in this paper concerning the general construction of configuration space representations of the Heisenberg algebra over an arbitrary configuration manifold, whether flat or curved, or parametrised in terms of curvilinear coordinates, raises a series of comments and further issues. Quantum dynamics requires the definition of a Riemannian metric structure on configuration space, whose determinant directly specifies the normalisation of position eigenstates in order to ensure proper covariant properties of the Heisenberg algebra representation under diffeomorphisms of the configuration manifold, namely changes of coordinate parametrisations. Such a property is certainly a necessary requirement for a consistent physical description of any system. In addition, due to the local arbitrariness in the phase of position eigenstates, a flat U(1) bundle is always associated to any such representation of the Heisenberg algebra. In the case of a simply connected manifold, this flat U(1) bundle may always be trivialized globally over the entire configuration manifold, thereby corresponding to the ordinary trivial representation of the Heisenberg algebra. However, for configuration spaces of non trivial mapping class group $\pi_1(M)$, an infinity of inequivalent representations becomes possible, being labelled by the non trivial holonomies of the flat U(1) bundle around the non contractible cycles in the configuration manifold. Based on these general conclusions, we have also shown how to construct the configuration space wave functions of the momentum eigenstates, as well as representations of quantum amplitudes in terms of discretized path integrals over phase space. Finally, through two simple examples borrowed from non relativistic quantum mechanics, we have demonstrated that the general approach developed here is consistent, and does indeed possess the different features which it is advocated to achieve.

In principle, the construction developed in this paper should lead to self-adjoint position and momentum observables, provided the necessary restrictions on states which were considered are met. This specific issue should thus certainly be confronted with the usual discussion of self-adjoint extensions of hermitian differential operators[2] in terms of von Neumann's deficiency indices.

Especially with the second example of the spherical harmonic oscillator on a punctured plane, it should be clear that a simple physical picture may be developed for the non trivial holonomies associated to the non trivial representations of the Heisenberg algebra. Indeed, the holonomy parametrized by λ may also be seen as nothing else than a Aharonov-Bohm (AB) magnetic flux line[9] piercing the plane at its origin and in whose vector potential the harmonic oscillator is forced to oscillate. In particular for the choice $(\mu, \nu) = (0, 0)$, the solutions constructed in this paper should reproduce, in the limit $\omega = 0$, well established results[9, 10] for particle scattering off an infinitely thin AB flux line[22]. Hence more generally, the non trivial holonomies associated to non trivial representations of the Heisenberg algebra may be regarded as being due to specific AB flux lines passing through the different holes in configuration space which are characterized by the first homotopy group $\pi_1(M)$ of that space. For example, the representations of the Heisenberg algebra in the case of a particle moving on a circle (see Ref.[11]) are labelled in terms of a U(1) holonomy which may be viewed as a AB flux line passing through the center of that circle. The same picture applies in the case of a 2-torus with two AB flux lines, one passing through the hole of the torus, and the other closed onto itself and lying inside the volume of the torus. More generally, one may imagine that for a configuration space with a complicated maze of holes, a whole network of intertwined open and closed AB flux lines threading these holes defines all the unitarily inequivalent representations of the Heisenberg algebra associated to that configuration space, the latter then viewed as being embedded in some higher dimensional manifold. *A priori*, such a picture may well apply in the case of the modular spaces of non-abelian Yang-Mills gauge theories or theories of gravity, the topology of these spaces being particularly rich, and presumably at the origin of the non-perturbative dynamics in such systems.

Another question which arises is obviously the possible relationship between the flat U(1) bundle which is a characteristic of representations of the Heisenberg algebra, and the well known Berry phase. It would certainly prove very clarifying to identify the connection between these two aspects of quantised systems with multiply connected configuration spaces.

The latter aspect also raises the issue of the possible realisation within the general and abstract context of geometric quantisation[12] of the present approach to the quantisation on arbitrary manifolds. In particular, our results are specifically reminiscent of those of Ref.[13], in which the general methods of induced representations are applied to the quantisation on homogeneous coset spaces G/H defined by Lie groups G and H. Indeed, in the latter case, a H

[22]Note how the parameter ω may be regarded as a regularisation parameter for the non normalisable states of a free particle. It may thus prove useful in the context of AB scattering and/or bound state issues, to determine quantities for finite ω, and let ω vanish only in the end result.

bundle arises in the classification of the possible quantum realisations of such systems.

As a matter of fact, the differences between our approach and that of Ref.[13] are the following. First, our approach is applicable to whatever curved or flat manifold, independently of any specific symmetries that manifold may possess. In contradistinction, coset manifolds G/H possess additional Killing vectors which generate symmetries of the manifold, and whose existence is essential to the construction of Ref.[13]. Second, our approach considers the classification of representations of the Heisenberg algebra only, whereas that of Ref.[13] constructs representations of a larger algebra which also includes the symmetry algebra induced by the Killing vectors of the manifold (see for example Ref.[11]). From that point of view, it should be clear that our approach could be extended to such manifolds as well, and lead to the same conclusion that inequivalent representations would be parametrised by some principal bundle whose base is the configuration space and whose fiber is related to a gauge connection in the algebra of the symmetry group of the manifold. Indeed, in the same manner as in Ref.[13], configuration space wave functions would become vector valued into some representation space of the symmetry group of the manifold, while the rôle of the flat U(1) bundle in the case of the Heisenberg algebra would be extended to some gauge connection in the relevant symmetry group of the manifold. Hence, even though our approach is at the same time more general (arbitrary manifold) and more particular (representations of the Heisenberg algebra only), it seems fair to conclude that our results and those of Ref.[13] are consistent and complementary, and provide constructions of quantum systems in curvilinear coordinates or on curved manifolds in less or more abstract mathematical terms.

Applications of the considerations of this paper should clearly be numerous and of varying degrees of difficulty. The case of modular spaces of Yang-Mills gauge theories has already been mentioned, as the ultimate example of multiply-connected configurations spaces. Far less ambitious but yet important would the quantisation of systems whose configuration space is a curved manifold[14], or even more simply, some flat euclidean space but parametrised by curvilinear coordinates rather than cartesian ones. As we have seen, even though only the trivial representation of the Heisenberg algebra is then relevant in the latter instance—the mapping class group of such manifolds being trivial—, nevertheless a correct treatment of the factor $g(q) = \det g_{\alpha\beta}(q)$ determined from the metric tensor is essential in obtaining the correct canonical quantisation of the system in curvilinear coordinates, and in particular in defining self-adjoint representations of the position and momentum operators on the configuration manifold. One class of systems where this latter point may be particularly relevant is the parametrisation of the N-body problem in terms of hyperspherical coordinates[15].

8 Acknowledgments

Profs. J.C. Alvarez, J.-P. Antoine, J.L. Lucio-Martinez and A. Magnus are acknowledged for useful discussions. This work has been partially supported under the terms of an agreement between the CONACyT (Mexico) and the FNRS (Belgium).

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